APPLIED MATHEMATICS 1 Question Bank

📢 Important Note Regarding Applied Mathematics Question Bank

Dear First-Year Engineering Students,

We are pleased to provide you with a carefully prepared **Applied Mathematics-I Question Bank** based on the analysis of past year question papers (PYQs) adhering to the NEP 2020 syllabus.

Please read this **caution** carefully:

1. **Reference Material Only:** This Question Bank is solely for reference and practice purposes.
2. **Focus on Frequency:** It is designed to highlight the most frequently tested concepts and question structures from the past examinations.
3. **No Guarantee:** There is absolutely no guarantee that the questions in the upcoming examination will be drawn exclusively from this set.
4. **Comprehensive Preparation is Key:** High scores are achieved by mastering the entire syllabus (Modules 1 through 6), understanding the underlying concepts, and practising both derivations and numerical problems.
5. **Do check PYQs:** For questions that are not given here, refer to the full past papers.

Our recommendation is to **use this Question Bank as a guide to:**

- **Identify High-Priority Topics:** Specifically Leibnitz Theorem, Euler's Theorem on Homogeneous Functions, Consistency of Linear Equations, PAQ Form, and Gauss Seidel Method.
- **Master the Standard Proofs:** Focus on the 6-mark theoretical proofs (e.g., Leibnitz theorem statements, properties of Unitary matrices).
- **Practice Numericals:** Ensure speed and accuracy in algorithmic topics like Newton Raphson and Matrix Rank reduction.

Do not limit your study to these questions alone. Ensure you cover all the concepts listed in the Applied Mathematics syllabus for complete and thorough preparation.

Best of luck with your studies!

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List of IMP/Frequent Topics Module Wise:

MODULE 1: COMPLEX NUMBERS

1. Roots of Complex Numbers
2. Expansion of Trignometric Functions
3. De Moivre's Applications

MODULE 2:HYPERBOLIC FUNCTIONS & LOGARITHMS OF COMPLEX NUMBERS

1. Logarithm of Complex Numbers
2. Inverse Hyperbolic
3. Separation of Real/Imaginary

MODULE 3:PARTIAL DIFFERENTIATION

1. Euler's Theorem Verification
2. Composite Functions
3. Proof of Euler's Theorem
4. Partial Differentiation Identity

MODULE 4: APPLICATION OF PARTIAL DIFFERENTIATION & SUCCESSIVE DIFFERENTIATION

1. Leibnitz Theorem
2. Maxima and Minima
3. n^{th} Derivative.

MODULE 5: MATRICES

1. Consistency of Equations
2. PAQ Form / Normal Form
3. Unitary Matrix
4. Inverse of Matrix

MODULE 6: NUMERICAL SOLUTIONS OF TRANSCENDENTAL EQUATIONS AND SYSTEM

1. Gauss Seidel / Jacobi
2. Newton Raphson
3. Iterative Formula derivation

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__List of IMP/Frequent Questions Module Wise:__

MODULE 1: COMPLEX NUMBERS

1. Find all values of $(1 + i)^{1/3}$.
2. Find all the roots of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ hence find the continued product of the roots.
3. Prove that $cos^6\theta = \frac{1}{32}(cos6\theta + 6cos4\theta + 15cos2\theta + 10)$.
4. Expand $sin^5\theta cos^3\theta$ in a series of sines of multiples of θ.
5. If $cos\alpha + cos\beta + cos\gamma = 0$ and $sin\alpha + sin\beta + sin\gamma = 0$, prove that... $sin3\alpha + sin3\beta + sin3\gamma = 3sin(\alpha + \beta + \gamma)$.
6. Prove that $\sqrt{1 + cosec(\theta/2)} = \left(1 - e^{i\theta}\right)^{-1/2} + \left(1 - e^{-i\theta}\right)^{-1/2}$.

MODULE 2:HYPERBOLIC FUNCTIONS & LOGARITHMS OF COMPLEX NUMBERS

7. Prove that $log\left(e^{i\alpha} + e^{i\beta}\right) = log[2cos\left(\frac{\alpha-\beta}{2}\right)] + i\left(\frac{\alpha+\beta}{2}\right)$.
8. If $tan[log(x + iy)] = a + ib$, prove that $tan[log\left(x^2 + y^2\right)] = \frac{2a}{1-a^2-b^2}$.
9. Prove that $sech^{-1}(sin\theta) = log\left(cot\frac{\theta}{2}\right)$.
10. Prove that: $tanh\left(log\sqrt{3}\right) = 0.5$.
11. If $\alpha + i\beta = tanh\left(x + \frac{i\pi}{4}\right)$, prove that $\alpha^2 + \beta^2 = 1$.

MODULE 3:PARTIAL DIFFERENTIATION

12. If $u = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then prove that $x^2\frac{\partial^2u}{\partial x^2} + 2xy\frac{\partial^2u}{\partial x\partial y} + y^2\frac{\partial^2u}{\partial y^2} =- \frac{sinucos2u}{4cos^3u}$.
13. If $H = f(y - z, z - x, x - y)$, then prove that $\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0$.
14. If $u = f\left(e^{x-y}, e^{y-z}, e^{z-x}\right)$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.
15. State & prove Euler's theorem on homogeneous functions with two independent variables.
16. If $u = \left(1 - 2xy + y^2\right)^{-1/2}$, prove that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3$.
17. If $u = 2(ax + by)^2 - k\left(x^2 + y^2\right)$ and $a^2 + b^2 = k$, find $\frac{\partial^2u}{\partial x^2} + \frac{\partial^2u}{\partial y^2}$.

MODULE 4: APPLICATION OF PARTIAL DIFFERENTIATION & SUCCESSIVE DIFFERENTIATION

18. If $y = e^{tan^{-1}x}$ then prove that $\left(1 + x^2\right)y_{n+2} + [2(n + 1)x - 1]y_{n+1} + n(n + 1)y_n = 0$.
19. If $y = acos(logx) + bsin(logx)$, prove that $x^2y_{n+2} + (2n + 1)xy_{n+1} + \left(n^2 + 1\right)y_n = 0$.

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20. Discuss the maxima and minima of $\left(x^2 + y^2 + 8x + 6y + 6\right)$.
21. Examine the function $f(x, y) = y^2 + 4xy + 3x^2 + x^3$ for extreme values.
22. Find n^{th} derivative of $y = \frac{x}{1+3x+2x^2}$.
23. Find n^{th} derivative of $\frac{2}{(x-1)(x-2)(x-3)}$.

MODULE 5: MATRICES

24. Investigate for what values of λ and μ the system of linear equations $x + y + z = 6$; $x + 2y + 3z = 10$; $x + 2y + \lambda z = \mu$ will have (i) No solution, (ii) Unique solution, (iii) Infinite number of solutions.
25. Find nonsingular matrices P & Q such that PAQ is in normal form and hence find rank of the matrix A for following matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$.
26. Reduce the following matrix to normal form and find the rank of matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$
27. Show that the matrix $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary and hence find A^{-1}.
28. Find a, b, c & A^{-1} if $A = \frac{1}{9}\begin{bmatrix} 8 & -4 & a \\ 1 & 4 & b \\ 4 & 7 & c \end{bmatrix}$ is orthogonal.
29. Find non-singular matrices P & Q such that PAQ is in normal form & hence find rank of the matrix A and obtain A^{-1}... $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$.

MODULE 6: NUMERICAL SOLUTIONS OF TRANSCENDENTAL EQUATIONS AND SYSTEM

30. Apply the Gauss Seidel method, to solve the following system of linear equations up to the two iterations. $x + 10y + z = 51, 15x + 3y - 2z = 85, x - 2y + 8z = 5$.
31. Apply Gauss-Seidel iteration method to solve the following linear equations up to the three iterations. $20x + y - 2z = 17$, $3x + 20y - z =- 18$, $2x - 3y + 20z = 25$..
32. Using Newton-Raphson Method, find the real root of $x^3 - 2x - 5 = 0$ correct to three decimal places.
33. Using Newton Raphson's Method... find an iterative formula for $\sqrt[3]{150}$.

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Module 1

1. Find all values of $(1 + i)^{1/3}$.
OR

2. Find all the roots of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ hence find the continued product of the roots.

Sol: **for 2nd**

Sol. : Let $\frac{1}{2} = r\cos\theta$ and $\frac{\sqrt{3}}{2} = r\sin\theta$.

$$\therefore \quad r^2(\cos^2\theta + \sin^2\theta) = \frac{1}{4} + \frac{3}{4} = 1 \qquad \therefore \quad r = 1$$

$$\therefore \quad \cos\theta = \frac{1}{2} \text{ and } \sin\theta = \frac{\sqrt{3}}{2} \qquad \therefore \quad \theta = \frac{\pi}{3}$$

$$\therefore \quad \left\{\frac{1}{2} + i\frac{\sqrt{3}}{2}\right\}^{3/4} = \left\{\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3\right\}^{1/4} = (\cos\pi + i\sin\pi)^{1/4}$$

$$= [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/4}$$

$$= \cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}.$$

Putting $k = 0, 1, 2, 3$, we get the four roots as,

$$\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right), \left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right).$$

Their continued product,

$$= \cos\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right)$$

$$= \cos 4\pi + i\sin 4\pi = 1.$$

3. Prove that $cos^6\theta = \frac{1}{32}(cos6\theta + 6cos4\theta + 15cos2\theta + 10)$.

Sol:

Step 1: Express $cos\theta$ in terms of complex numbers Let $z = cos\theta + isin\theta$. Then $\frac{1}{z} = cos\theta - isin\theta$. Using the property of addition:

$$z + \frac{1}{z} = 2cos\theta$$

We can generalize this for any power n using De Moivre's Theorem:

$$z^n + \frac{1}{z^n} = 2cosn\theta$$

Step 2: Raise to the power of 6 We start with the equation $2cos\theta = z + \frac{1}{z}$. Raise both sides to the power of 6:

$$(2cos\theta)^6 = \left(z + \frac{1}{z}\right)^6$$

$$64cos^6\theta = \left(z + \frac{1}{z}\right)^6$$

Step 3: Expand using Binomial Theorem Using $(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$:

$$\left(z + \frac{1}{z}\right)^6 = z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) + 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) + 6z\left(\frac{1}{z^5}\right) + \frac{1}{z^6}$$

Simplify the powers of z:

$$= z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$$

Step 4: Group like terms Group terms with the same powers of z and $1/z$:

$$= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$$

Step 5: Substitute trigonometric forms back Use the identity $z^n + \frac{1}{z^n} = 2cosn\theta$:

$$64cos^6\theta = (2cos6\theta) + 6(2cos4\theta) + 15(2cos2\theta) + 20$$

Step 6: Simplify Factor out 2 from the Right Hand Side (R.H.S.):

$$64cos^6\theta = 2[cos6\theta + 6cos4\theta + 15cos2\theta + 10]$$

Divide both sides by 64:

$$cos^6\theta = \frac{2}{64}(cos6\theta + 6cos4\theta + 15cos2\theta + 10)$$

$$cos^6\theta = \frac{1}{32}(cos6\theta + 6cos4\theta + 15cos2\theta + 10)$$

Hence Proved.

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4. Expand $sin^5\theta cos^3\theta$ in a series of sines of multiples of θ.

Sol:

Sol. : As above we can write

$$(2\,i\sin\theta)^5(2\cos\theta)^3=\left(x-\frac{1}{x}\right)^5\left(x+\frac{1}{x}\right)^3$$

$$=\left(x-\frac{1}{x}\right)^2\left(x-\frac{1}{x}\right)^3\left(x+\frac{1}{x}\right)^3=\left(x-\frac{1}{x}\right)^2\left(x^2-\frac{1}{x^2}\right)^3$$

$$=\left(x^2-2+\frac{1}{x^2}\right)\left(x^6-3x^2+\frac{3}{x^2}-\frac{1}{x^6}\right)$$

$$=x^8-3x^4+3-\frac{1}{x^4}-2x^6+6x^2-\frac{6}{x^2}+\frac{2}{x^6}+x^4-3+\frac{3}{x^4}-\frac{1}{x^8}$$

$$=\left(x^8-\frac{1}{x^8}\right)-2\left(x^6-\frac{1}{x^6}\right)-2\left(x^4-\frac{1}{x^4}\right)+6\left(x^2-\frac{1}{x^2}\right)$$

$$\therefore\quad 2^8 i\sin^5\theta\cos^3\theta=2i\sin 8\theta-2\cdot 2i\sin 6\theta-2\cdot 2i\sin 4\theta+6\cdot 2i\sin 2\theta$$

$$\therefore\quad \sin^5\theta\cos^3\theta=\frac{1}{128}[\sin 8\theta-2\sin 6\theta-2\sin 4\theta+6\sin 2\theta]$$

5. If $cos\alpha + cos\beta + cos\gamma = 0$ and $sin\alpha + sin\beta + sin\gamma = 0$, prove that... $sin3\alpha + sin3\beta + sin3\gamma = 3sin(\alpha + \beta + \gamma)$.

OR

6. Prove that $\sqrt{1 + cosec(\theta/2)} = \left(1 - e^{i\theta}\right)^{-1/2} + \left(1 - e^{-i\theta}\right)^{-1/2}$.

Sol:

for 1 st

Sol. : Let us use a, b, c as in Ex. 2, then $a+b+c=0$.

As seen in Ex. 1, above $a^3+b^3+c^3=3\,abc$.

[Or Since $a+b+c=0$, $(a+b)=(-c)$

$\therefore\ (a+b)^3=-c^3$ $\qquad\therefore\ a^3+b^3+3ab\,(a+b)=-c^3$

$\therefore\ a^3\ b^3+3ab\,(-c)=-c^3$ $\qquad\therefore\ a^3+b^3+c^3=3abc.$]

$$\therefore\quad (\cos\alpha+i\sin\alpha)^3+(\cos\beta+i\sin\beta)^3+(\cos\alpha+i\sin\gamma)^3$$
$$=3\,(\cos\alpha+i\sin\alpha)\,(\cos\beta+i\sin\beta)\,(\cos\gamma+i\sin\gamma)$$

$$\therefore\quad (\cos 3\alpha+\cos 3\beta+\cos 3\gamma)+i\,(\sin 3\alpha+\sin 3\beta+\sin 3\gamma)$$
$$=3\,[\,\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)\,]$$

Equating the real and imaginary parts, we get the required results.

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MODULE 2

1. Prove that $log\left(e^{i\alpha} + e^{i\beta}\right) = log[2cos\left(\frac{\alpha-\beta}{2}\right)] + i\left(\frac{\alpha+\beta}{2}\right)$.

(Logarithm of Complex Numbers)

Sol:

Step 1: Manipulate the term inside the logarithm. Consider the expression $Z = e^{i\alpha} + e^{i\beta}$ .We factor out $e^{i\left(\frac{\alpha+\beta}{2}\right)}$ from both terms to create symmetric exponents.

$$e^{i\alpha} + e^{i\beta} = e^{i\frac{\alpha+\beta}{2}} \cdot \left[\frac{e^{i\alpha}}{e^{i\frac{\alpha+\beta}{2}}} + \frac{e^{i\beta}}{e^{i\frac{\alpha+\beta}{2}}}\right]$$

Using exponent rules ($\frac{e^{a}}{e^{b}} = e^{a-b}$):First term: $i\alpha - i\frac{\alpha+\beta}{2} = i\frac{2\alpha-\alpha-\beta}{2} = i\frac{\alpha-\beta}{2}$Second term: $i\beta - i\frac{\alpha+\beta}{2} = i\frac{2\beta-\alpha-\beta}{2} =- i\frac{\alpha-\beta}{2}$So, the expression becomes:

$$e^{i\alpha} + e^{i\beta} = e^{i\frac{\alpha+\beta}{2}}\left[e^{i\frac{\alpha-\beta}{2}} + e^{-i\frac{\alpha-\beta}{2}}\right]$$

Step 2: Apply Euler's Exponential FormWe know that $cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, which implies $e^{i\theta} + e^{-i\theta} = 2cos\theta$.Here, $\theta = \frac{\alpha-\beta}{2}$.Substituting this into our equation:

$$e^{i\alpha} + e^{i\beta} = e^{i\frac{\alpha+\beta}{2}} \cdot \left[2cos\left(\frac{\alpha-\beta}{2}\right)\right]$$

Step 3: Take the Logarithm Now, take log on both sides:

$$log\left(e^{i\alpha} + e^{i\beta}\right) = log\left[2cos\left(\frac{\alpha-\beta}{2}\right) \cdot e^{i\frac{\alpha+\beta}{2}}\right]$$

Using the property $log(AB) = logA + logB$:

$$= log\left[2cos\left(\frac{\alpha-\beta}{2}\right)\right] + log\left[e^{i\frac{\alpha+\beta}{2}}\right]$$

Using the property $log\left(e^{x}\right) = x$:

$$= log\left[2cos\left(\frac{\alpha-\beta}{2}\right)\right] + i\left(\frac{\alpha+\beta}{2}\right)$$

Hence Proved.

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2. If $tan[log(x + iy)] = a + ib$, prove that $tan[log(x^2 + y^2)] = \frac{2a}{1-a^2-b^2}$.

Sol. : We have $\tan[\log(x+iy)] = a+ib$ $\quad \therefore \log(x+iy) = \tan^{-1}[a+ib]$

Let $\tan^{-1}(a+ib) = \alpha + i\beta$ .... (i)

$\therefore \log(x+iy) = \alpha + i\beta \qquad \therefore \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x} = \alpha + i\beta$

Equating real parts, we have

$$\log\sqrt{x^2+y^2} = \alpha \quad i.e. \quad \frac{1}{2}\log(x^2+y^2) = \alpha$$

$\therefore \log(x^2+y^2) = 2\alpha$ .... (ii)

We now find the real part α of (i)

$$\tan 2\alpha = \tan[(\alpha+i\beta)+(\alpha-i\beta)] = \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta)\tan(\alpha-i\beta)} \quad \text{..................... (iii)}$$

Since $\tan^{-1}(a+ib) = \alpha + i\beta$

$\therefore \tan(\alpha+i\beta) = a+ib$ and $\tan(\alpha-i\beta) = a-ib$

Hence, from (iii), we get,

$$\tan 2\alpha = \frac{(a+ib)+(a-ib)}{1-(a+ib)+(a-ib)} = \frac{2a}{1-a^2-b^2} \quad \text{..................... (iv)}$$

Hence, by putting the values of 2α from (ii) in (iv),

$$\therefore \tan[\log(x^2+y^2)] = \frac{2a}{1-a^2-b^2} \text{ when } a^2+b^2 \neq 1. \qquad [\text{By (iv)}]$$

3. Prove that $sech^{-1}(sin\theta) = log\left(cot\frac{\theta}{2}\right)$.

Sol:

Sol. : Let $\sec h^{-1}(\sin\theta) = x \quad \therefore \sin\theta = \sec hx$

$$\therefore \sin\theta = \frac{1}{\cos hx} = \frac{2}{e^x+e^{-x}} = \frac{2e^x}{e^{2x}+1}$$

$$\therefore \sin\theta \cdot e^{2x} - 2e^x + \sin\theta = 0$$

This is a quadratic in e^x.

$$\therefore e^x = \frac{2 \pm \sqrt{4-4\sin^2\theta}}{2\sin\theta} = \frac{1 \pm \cos\theta}{\sin\theta}$$

$$= \frac{1+\cos\theta}{\sin\theta} = \frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \cot\frac{\theta}{2}$$

$$\therefore x = \log\left(\cot\frac{\theta}{2}\right)$$

$$\therefore \sec h^{-1}(\sin\theta) = \log\left(\cot\frac{\theta}{2}\right)$$

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4. Prove that: $tanh\left(log\sqrt{3}\right) = 0.5$.

Sol.

Step 1: Use the definition of $tanhx$We know that:

$$tanhx = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Step 2: Substitute $x = log\sqrt{3}$

$$tanh\left(log\sqrt{3}\right) = \frac{e^{log\sqrt{3}} - e^{-log\sqrt{3}}}{e^{log\sqrt{3}} + e^{-log\sqrt{3}}}$$

Step 3: Simplify the exponential terms Recall that $e^{loga} = a$ and $e^{-loga} = e^{log(1/a)} = \frac{1}{a}$ .Here, let $a = \sqrt{3}$.

$$e^{log\sqrt{3}} = \sqrt{3}e^{-log\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Substitute these back into the fraction:

$$tanh\left(log\sqrt{3}\right) = \frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{\sqrt{3} + \frac{1}{\sqrt{3}}}$$

Step 4: Algebraic simplification Multiply numerator and denominator by $\sqrt{3}$:

$$= \frac{\left(\sqrt{3}\right)^2 - 1}{\left(\sqrt{3}\right)^2 + 1} = \frac{3-1}{3+1}$$

$$= \frac{2}{4} = 0.5$$

Hence Proved.

5. If $\alpha + i\beta = tanh\left(x + \frac{i\pi}{4}\right)$, prove that $\alpha^2 + \beta^2 = 1$.

Sol:

Sol. : To prove the required result, we have to separate real part α from imaginary part β.

$$\alpha + i\beta = \frac{\sin h[x + (i\pi/4)]}{\cos h[x + (i\pi/4)]}$$

$$= \frac{2\sin h[x + (i\pi/4)]}{2\cos h[x + (i\pi/4)]} \cdot \frac{\cos h[x - (i\pi/4)]}{\cos h[x - (i\pi/4)]}$$

$$= \frac{\sin h 2x + \sin h(i\pi/2)}{\cos h 2x + \cos h(i\pi/2)} \quad \text{[By (d), page 3-5]}$$

$$= \frac{\sin h 2x + i\sin(\pi/2)}{\cos h 2x + \cos(\pi/2)} = \frac{\sin h 2x + i}{\cos h 2x}$$

Equating real and imaginary parts.

$$\therefore \quad \alpha = \frac{\sin h 2x}{\cos h 2x}, \quad \beta = \frac{1}{\cos h 2x}$$

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$$\therefore \quad \frac{\sqrt{2}}{\cosh x + i\sinh x} = u + iv \qquad \therefore \quad \frac{\sqrt{2}\,(\cosh x - i\sinh x)}{\cosh^2 x + \sinh^2 x} = u + iv$$

$$\therefore \quad \frac{\sqrt{2}\,(\cosh x - i\sinh x)}{\cosh 2x} = u + iv$$

Equating real and imaginary parts,

$$\therefore \quad u = \frac{\sqrt{2}\cosh x}{\cosh 2x}, \quad v = -\frac{\sqrt{2}\sinh x}{\cosh 2x}$$

$$\therefore \quad u^2 + v^2 = \frac{2}{\cosh^2 2x}(\cosh^2 x + \sinh^2 x) = \frac{2\cosh 2x}{\cosh^2(2x)} = \frac{2}{\cosh 2x}$$

$$\therefore \quad (u^2 + v^2)^2 = \left(\frac{2}{\cosh 2x}\right)^2 = \frac{4}{\cosh^2 2x} \quad \text{.......................} (1)$$

Also $$u^2 - v^2 = \frac{2}{\cosh^2 2x}(\cosh^2 x - \sinh^2 x) = \frac{2}{\cosh^2 2x} \quad \text{.......................} (2)$$

From (1) and (2), $(u^2 + v^2)^2 = 2\,(u^2 - v^2)$.

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MODULE 3

1. If $u = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{sinucos2u}{4cos^3 u}$.

Sol:

(M.U. 1986, 91, 2

Sol. : Note that u is not a homogeneous function of x and y.

Let $z = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = f(u) = F(X, Y)$, say.

Putting $X = xt$, $Y = yt$, we get,

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$$F(X,Y) = \frac{X+Y}{\sqrt{X}+\sqrt{Y}} = \frac{xt+yt}{\sqrt{xt}+\sqrt{yt}}$$

$$= \frac{t}{\sqrt{t}} \cdot \frac{(x+y)}{\sqrt{x}+\sqrt{y}} = t^{1/2} \cdot \frac{x+y}{\sqrt{x}+\sqrt{y}} = t^{1/2} F(x,y)$$

Thus, $z = \sin u$ is a homogeneous function of x, y of degree 1/2.

Hence, by the above corollary,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 y}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u) \cdot [g'(u) - 1]$$

where, $g(u) = n \cdot \frac{f(u)}{f'(u)} = \frac{1}{2} \cdot \frac{\sin u}{\cos u} = \frac{1}{2}\tan u.$

$$\therefore \quad g'(u) = \frac{1}{2}\sec^2 u$$

$$\therefore \quad g(u)[g'(u) - 1] = \frac{1}{2}\tan u \cdot \left[\frac{1}{2}\sec^2 u - 1\right]$$

$$= \frac{1}{4}\tan u\left[\frac{1-2\cos^2 u}{\cos^2 u}\right] = \frac{1}{4\cos^3 u} \cdot \sin u(-\cos 2u)$$

$$= -\frac{\sin u \cos 2u}{4\cos^3 u}.$$

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2. If $u = f(x-y, y-z, z-x)$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Sol:

Let $X = x-y$, $Y = y-z$, $Z = z-x$, then $u = f(X,Y,Z)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x}$$

$$= \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z}$$

Similarly, $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial Y} - \frac{\partial u}{\partial X}$.

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial Z} - \frac{\partial u}{\partial Y}$$

Adding the three result

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

3. If $u = f\left(e^{x-y}, e^{y-z}, e^{z-x}\right)$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Sol:

Let $X = e^{x-y}$, $Y = e^{y-z}$, $Z = e^{z-x}$ then $u = f(X,Y,Z)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x}$$

$$= \frac{\partial u}{\partial X} e^{x-y}(1) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z} e^{z-x}(-1)$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} e^{x-y} - \frac{\partial u}{\partial Z} e^{z-x} \quad \text{—— (1)}$$

Now $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y}$

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$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} e^{x-y}(-1) + \frac{\partial u}{\partial Y}\cdot e^{y-z}(1) + \frac{\partial u}{\partial Z}(0)$$

And $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z}$

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$$= -\frac{\partial u}{\partial X} \cdot e^{x-y} + \frac{\partial u}{\partial Y} \cdot e^{y-z} \quad \text{—— (2)}$$

And $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial z}$$

$$= \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y} e^{y-z}(-1) + \frac{\partial u}{\partial Z} e^{z-x}(1)$$

$$\therefore \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial Y} e^{y-z} + \frac{\partial u}{\partial Z} \cdot e^{z-x} \quad \text{—— (3)}$$

Adding (1), (2) and (3), we get

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0}$$

4. State & prove Euler's theorem on homogeneous functions with two independent variables.

Sol.

Statement of Euler's Theorem

If z is a homogeneous function of two variables x and y of degree n, then:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Proof:

Step 1: Definition of Homogeneity Let $z = f(x, y)$ be the given function. Since it is a homogeneous function of degree n, by definition, if we scale the variables by a parameter t :

$$f(xt, yt) = t^n f(x, y) \quad ...... \ (i)$$

Let $X = xt$ and $Y = yt$. Then the equation becomes:

$$f(X, Y) = t^n f(x, y)$$

Step 2: Differentiate with respect to t We differentiate the left-hand side (L.H.S.) of equation (i) with respect to t using the chain rule:

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$$\frac{\partial}{\partial t} f(X, Y) = \frac{\partial f}{\partial X} \cdot \frac{\partial X}{\partial t} + \frac{\partial f}{\partial Y} \cdot \frac{\partial Y}{\partial t}$$

Since $X = xt$ and $Y = yt$:

1. $\frac{\partial X}{\partial t} = x$
2. $\frac{\partial Y}{\partial t} = y$

Substituting these back:

$$\frac{\partial f}{\partial t} = x\frac{\partial f}{\partial X} + y\frac{\partial f}{\partial Y}$$

Now, put $t = 1$. This implies $X = x$ and $Y = y$. The equation becomes:

$$\left[\frac{\partial f}{\partial t}\right]_{t=1} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \quad ...... \ (ii)$$

Step 3: Differentiate the R.H.S. Now, differentiate the right-hand side (R.H.S.) of equation (i) with respect to t:

$$\frac{\partial}{\partial t}\left[t^n f(x, y)\right] = nt^{n-1} f(x, y)$$

Now, put $t = 1$:

$$\left[\frac{\partial f}{\partial t}\right]_{t=1} = n(1)^{n-1} f(x, y) = nf(x, y) \quad ...... \ (iii)$$

Step 4: Equate both sides From (ii) and (iii), we equate the values of $\left[\frac{\partial f}{\partial t}\right]_{t=1}$:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

Since $z = f(x, y)$:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Hence Proved.

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5. If $u = \left(1 - 2xy + y^2\right)^{-1/2}$, prove that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3$.

Sol:

Sol. : Since $u = (1 - 2xy + y^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2y) = yu^3 \qquad \therefore \quad x\frac{\partial u}{\partial x} = xy\,u^3$$

Also, $$\frac{\partial u}{\partial y} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2x + 2y) = (x - y)u^3 \qquad \therefore \quad y\frac{\partial u}{\partial y} = (xy - y^2)u^3$$

$$\therefore \quad x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xy\,u^3 - xy\,u^3 + y^2u^3 = y^2u^3.$$

6. If $u = 2(ax + by)^2 - k\left(x^2 + y^2\right)$ and $a^2 + b^2 = k$, find $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

Sol. **For 1st**

Sol. : Since $u = (1 - 2xy + y^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2y) = yu^3 \qquad \therefore \quad x\frac{\partial u}{\partial x} = xy\,u^3$$

Also, $$\frac{\partial u}{\partial y} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2x + 2y) = (x - y)u^3 \qquad \therefore \quad y\frac{\partial u}{\partial y} = (xy - y^2)u^3$$

$$\therefore \quad x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xy\,u^3 - xy\,u^3 + y^2u^3 = y^2u^3.$$

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MODULE 4

1. If $y = e^{tan^{-1}x}$ then prove that $(1 + x^2)y_{n+2} + [2(n + 1)x - 1]y_{n+1} + n(n + 1)y_n = 0$.

Sol:

Step 1: Find the first derivative (y_1)

$$y = e^{\tan^{-1} x}$$

Differentiate with respect to x:

$$y_1 = e^{\tan^{-1} x} \cdot \frac{1}{1+x^2}$$

$$y_1 = y \cdot \frac{1}{1+x^2} \quad (\because e^{\tan^{-1} x} = y)$$

Cross-multiply:

$$(1+x^2)y_1 = y$$

Step 2: Find the second derivative (y_2) Differentiate both sides with respect to x again:

$$(1+x^2)y_2 + y_1(2x) = y_1$$

$$(1+x^2)y_2 + (2x-1)y_1 = 0 \quad \text{...... (i)}$$

Step 3: Apply Leibnitz's Theorem Differentiate equation (i) n times using Leibnitz's Theorem: $(uv)_n = uv_n + nc_1u_1v_{n-1} + nc_2u_2v_{n-2} + \dots$

- **Term 1:** $D^n[(1+x^2)y_2]$

$$= (1+x^2)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2}(2)y_n$$

$$= (1+x^2)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n$$

- **Term 2:** $D^n[(2x-1)y_1]$

$$= (2x-1)y_{n+1} + n(2)y_n$$

$$= (2x-1)y_{n+1} + 2ny_n$$

Step 4: Combine and Simplify Add the terms:

$$(1+x^2)y_{n+2} + [2nx + (2x-1)]y_{n+1} + [n(n-1) + 2n]y_n = 0$$

$$(1+x^2)y_{n+2} + [2x(n+1) - 1]y_{n+1} + [n^2 - n + 2n]y_n = 0$$

$$(1+x^2)y_{n+2} + [2(n+1)x - 1]y_{n+1} + n(n+1)y_n = 0$$

Hence Proved.

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2. If $y = acos(logx) + bsin(logx)$, prove that $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.

Sol:

Sol. : We have $y = a\cos(\log x) + b\sin(\log x)$

$$\therefore \quad y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$$

$$\therefore \quad xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Differentiating again w.r.t. x,

$$xy_2 + y_1 = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$$

$$\therefore \quad x^2y_2 + xy_1 + y = 0$$

Applying Leibnitz's Theorem to each term, we get

$$x^2y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2!}(2)y_n + [xy_{n+1} + n(1)y_n] + y_n = 0$$

$$\therefore \quad x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2 - n + n + 1)y_n = 0$$

$$\therefore \quad x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$$

3. Discuss the maxima and minima of $(x^2 + y^2 + 8x + 6y + 6)$.

Sol:

Sol. : We have $f(x, y) = x^2 + y^2 + 8x + 6y + 6.$

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Step I : $f_x = 2x + 8,\quad f_y = 2y + 6;\quad f_{xx} = 2,\quad f_{xy} = 0,\quad f_{yy} = 2.$

Step II : We now solve $f_x = 0,\ f_y = 0$

$\therefore\ 2x + 8 = 0 \quad \therefore\ x = -4; \quad 2y + 6 = 0 \quad \therefore\ y = -3$

$\therefore\ (-4, -3)$ is a stationary point.

Step III : For $x = -4,\ y = -3,\quad r = f_{xx} = 2,\quad s = f_{xy} = 0,\quad t = f_{yy} = 2.$

$\therefore\ rt - s^2 = 4 - 0 = 4 > 0.$ And $r = f_{xx} = 2 > 0$

$\therefore\ (-4, -3)$ is a minima.

The minimum value of $f(x, y) = 16 + 9 - 32 - 18 + 6 = -19.$

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4. Examine the function $f(x, y) = y^2 + 4xy + 3x^2 + x^3$ for extreme values.

Sol:

Sol. : We have $f(x, y) = y^2 + 4xy + 3x^2 + x^3$

Step I : $f_x = 4y + 6x + 3x^2$, $f_y = 2y + 4x$, $f_{xx} = 6 + 6x$, $f_{xy} = 4$, $f_{yy} = 2$.

Step II : We now solve $f_x = 0$, $f_y = 0$.

$\therefore$ $4y + 6x + 3x^2 = 0$ and $2y + 4x = 0$

Putting $2y = -4x$ in the first equation

$3x^2 - 2x = 0$ $\therefore$ $x(3x - 2) = 0$ $\therefore$ $x = 0$ or $x = 2/3$.

When $x = 0$, $y = 0$ and when $x = 2/3$, $y = -4/3$.

$\therefore$ (0, 0), (2/3, − 4/3) are stationary points.

Step III : (i) When $x = 0$, $y = 0$

$r = f_{xx} = 6$, $s = f_{xy} = 4$, $t = f_{yy} = 2$

$\therefore$ $rt - s^2 = 12 - 16 < 0$

$\therefore$ $f(x, y)$ is neither maximum nor minimum. It is a saddle point.

(ii) When $x = 2/3$, $y = -4/3$; $r = f_{xx} = 10$, $s = 4$, $t = 2$.

$\therefore$ $rt - s^2 = 20 - 16 = 4 > 0$

But $r = f_{xx} = 10 > 0$.

Hence, $f(x, y)$ is minimum at $x = 2/3$, $y = -4/3$.

The minimum value $= \frac{16}{9} - \frac{32}{9} + \frac{12}{9} + \frac{8}{27} = -\frac{4}{27}$.

5. Find n^{th} derivative of $y = \frac{x}{1+3x+2x^2}$.

Sol:

Step 1: Factorize the DenominatorThe denominator is $2x^2 + 3x + 1$.

$$2x^2 + 3x + 1 = 2x^2 + 2x + x + 1 = 2x(x + 1) + 1(x + 1) = (2x + 1)(x + 1)$$

So, $y = \frac{x}{(2x+1)(x+1)}$

Step 2: Resolve into Partial Fractions

Let $\frac{x}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1} x = A(x + 1) + B(2x + 1)$

Find B: Put $x = -1$

$$-1 = A(0) + B(2(-1) + 1) - 1 = B(-1) \Rightarrow B = 1$$

Find A: Put $x = -\frac{1}{2}$

$$-\frac{1}{2} = A\left(-\frac{1}{2} + 1\right) + B(0) - \frac{1}{2} = A\left(\frac{1}{2}\right) \Rightarrow A = -1$$

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Thus, $y = \frac{-1}{2x+1} + \frac{1}{x+1} y = (x + 1)^{-1} - (2x + 1)^{-1}$

Step 3: Find the n^{th} DerivativeApply the standard formula to each term:For $(x + 1)^{-1}$: Here $a = 1, b = 1$.

$$D^n[(x + 1)^{-1}] = \frac{(-1)^n n!(1)^n}{(x+1)^{n+1}}$$

For $(2x + 1)^{-1}$: Here $a = 2, b = 1$.

$$D^n[(2x + 1)^{-1}] = \frac{(-1)^n n!(2)^n}{(2x+1)^{n+1}}$$

Substituting these back:

$$y_n = \frac{(-1)^n n!}{(x+1)^{n+1}} - \frac{(-1)^n n! 2^n}{(2x+1)^{n+1}}$$

Final Answer: $y_n = (-1)^n n! \left[\frac{1}{(x+1)^{n+1}} - \frac{2^n}{(2x+1)^{n+1}}\right]$

6. Find n^{th} derivative of $\frac{2}{(x-1)(x-2)(x-3)}$.

Sol:

Sol. : Let $y = \frac{2}{(x-1)(x-2)(x-3)} = \frac{a}{(x-1)} + \frac{b}{(x-2)} + \frac{c}{(x-3)}$

$\therefore \quad 2 = a(x-2)(x-3) + b(x-1)(x-3) + c(x-1)(x-2)$

Putting $x = 1$, $\quad 2 = a(-1)(-2) \quad \therefore \; a = 1$

Putting $x = 2$, $\quad 2 = b(1)(-1) \quad \therefore \; b = -2$

Putting $x = 3$, $\quad 2 = c(2)(1) \quad \therefore \; c = 1$

$\therefore \quad y = \frac{1}{x-1} - \frac{2}{x-2} + \frac{1}{x-3}$

By result (4-A), we get,

$$y_n = (-1)^n \cdot n! \left[\frac{1}{(x-1)^{n+1}} - \frac{2}{(x-2)^{n+1}} + \frac{1}{(x-3)^{n+1}}\right].$$

(4-A)

$$\boxed{y_n = \frac{(-1)^n n!}{(x+b)^{n+1}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(4-A)}$$

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MODULE 5

1. Investigate for what values of λ and μ the system of linear equations $x + y + z = 6; x + 2y + 3z = 10; x + 2y + \lambda z = \mu$ will have (i) No solution, (ii) Unique solution, (iii) Infinite number of solutions.

Sol:

System of Equations:

1. $x + y + z = 6$
2. $x + 2y + 3z = 10$
3. $x + 2y + \lambda z = \mu$

Step 1: Form the Augmented Matrix $[A: B]$

$$[A: B] = [1\ 1\ 1\ 6\ 1\ 2\ 3\ 10\ 1\ 2\ \lambda\ \mu\ ]$$

Step 2: Reduce to Echelon Form Perform row operations to make the elements below the main diagonal zero.

Operation 1: $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\sim [1\ 1\ 1\ 6\ 0\ 1\ 2\ 4\ 0\ 1\ \lambda - 1\ \mu - 6\ ]$$

Operation 2: $R_3 \to R_3 - R_2$

$$\sim [1\ 1\ 1\ 6\ 0\ 1\ 2\ 4\ 0\ 0\ \lambda - 3\ \mu - 10\ ]$$

Now the matrix is in upper triangular form. We analyze the ranks of Matrix A (first 3 columns) and Augmented Matrix $[A: B]$.

Step 3: Analyze Conditions

Case (i): No Solution

For no solution, the system must be inconsistent. This happens when **Rank(A) ≠ Rank(A:B)**. Looking at the third row:

If $\lambda - 3 = 0$ (i.e., $\lambda = 3$), then Rank(A) becomes 2.
If $\mu - 10 \neq 0$ (i.e., $\mu \neq 10$), then Rank(A:B) remains 3.

Condition: $\lambda = 3$ and $\mu \neq 10$.

Case (ii): Unique Solution

For a unique solution, **Rank(A) = Rank(A:B) = Number of Unknowns (3)**. This requires the pivot in the third row of A to be non-zero.

$\lambda - 3 \neq 0 \Rightarrow \lambda \neq 3.$

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The value of μ does not matter (it can be any real number).

Condition: $\lambda \neq 3$ and μ can be any value.

Case (iii): Infinite Number of Solutions

For infinite solutions, the system must be consistent with **Rank(A) = Rank(A:B) < Number of Unknowns (3).** This happens if the entire third row becomes zero (Rank becomes 2).

$$\lambda - 3 = 0 \Rightarrow \lambda = 3.$$
$$\mu - 10 = 0 \Rightarrow \mu = 10.$$

Condition: $\lambda = 3$ and $\mu = 10$.

2. Find nonsingular matrices P & Q such that PAQ is in normal form and hence find rank of the matrix A for following matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$.

Sol:

Procedure: We perform row operations on A (and a pre-factor identity matrix I) to find P, and column operations on A (and a post-factor identity matrix I) to find Q, such that the final matrix is in the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

1. **Operation ($R_1 \to R_1 - R_2$):** Creates a leading 1.

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Update P: $R_1 \to R_1 - R_2$ on Identity.

2. **Operation ($R_2 \to R_2 - 2R_1$):** Clears the first column.

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

3. **Operation ($R_2 \leftrightarrow R_3$ then $R_2 \to -R_2$):** Brings the pivot -1 up and makes it positive.

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 4 \end{bmatrix}$$

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4. **Operation ($R_3 \rightarrow R_3 + 3R_2$):** Clears the second column.

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

5. **Operation ($C_3 \rightarrow C_3 + C_2$):** Clears the second row.

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Final Answer:

- **Normal Form:** $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- **Rank of A:** 3
- **Matrix P:** $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ -2 & 3 & -3 \end{bmatrix}$
- **Matrix Q:** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

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3. Reduce the following matrix to normal form and find the rank of matrix $A = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{bmatrix}$**.**

Sol:

Sol. : We have $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

By $\xrightarrow{\substack{R_2 - R_1 \\ R_3 - 3R_1}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}$ By $\xrightarrow{\substack{C_2 - C_1 \\ C_3 - C_1}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}$

By $\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ By $\xrightarrow{C_3 - C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

By $\xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ which is in normal form. $\therefore$ Rank = 2.

4. Show that the matrix $A = \frac{1}{\sqrt{3}} \begin{bmatrix} \mathbf{1} & \mathbf{1+i} \\ \mathbf{1-i} & \mathbf{-1} \end{bmatrix}$ **is unitary and hence find** A^{-1}**.**

Sol:

Sol. : Let us denote the given matrix by A.

$\therefore \quad A' = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \qquad \therefore \quad A^{\theta} = (\bar{A}') = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

Now, $A^{\theta} A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$\therefore \quad A^{\theta} A = \frac{1}{3} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Hence, A is unitary.

To Find A^{-1}

For a unitary matrix, the inverse is simply the conjugate transpose:

$$A^{-1} = A^{\theta}$$

From Step 1, we calculated A^{θ}:

$$A^{-1} = \frac{1}{\sqrt{3}} [1 \ 1 + i \ 1 - i \ - 1]$$

(Note: Since $A = A^{\theta}$, in this particular problem $A^{-1} = A$.)

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5. Find a, b, c & A^{-1} if $A = \frac{1}{9}\begin{bmatrix} 8 & -4 & a \\ 1 & 4 & b \\ 4 & 7 & c \end{bmatrix}$ is orthogonal.

Sol: For a matrix M to be orthogonal, $MM^T = I$. Let $A = \frac{1}{9}M$, where $M = [8\ -4\ a\ 1\ 4\ b\ 4\ 7\ c]$.

The condition $AA^T = I$ implies $\left(\frac{1}{9}M\right)\left(\frac{1}{9}M^T\right) = I$,

which simplifies to $MM^T = 81I$.

This gives us two conditions:

1. Sum of squares of each row equals 81:
   - $R_1: 8^2 + (-4)^2 + a^2 = 81 \Rightarrow 64 + 16 + a^2 = 81 \Rightarrow a^2 = 1 \Rightarrow a = \pm 1$
   - $R_2: 1^2 + 4^2 + b^2 = 81 \Rightarrow 1 + 16 + b^2 = 81 \Rightarrow b^2 = 64 \Rightarrow b = \pm 8$
   - $R_3: 4^2 + 7^2 + c^2 = 81 \Rightarrow 16 + 49 + c^2 = 81 \Rightarrow c^2 = 16 \Rightarrow c = \pm 4$
2. Dot product of any two distinct rows is 0:
   - $R_1 \cdot R_2 = 8(1) + (-4)(4) + ab = 0 \Rightarrow 8 - 16 + ab = 0 \Rightarrow ab = 8$
   - $R_1 \cdot R_3 = 8(4) + (-4)(7) + ac = 0 \Rightarrow 32 - 28 + ac = 0 \Rightarrow ac = -4$
   - $R_2 \cdot R_3 = 1(4) + 4(7) + bc = 0 \Rightarrow 4 + 28 + bc = 0 \Rightarrow bc = -32$

Solving for signs:

From $ab = 8$, a and b must have the same sign.

From $ac = -4$, a and c must have opposite signs.

Case 1: Let $a = 1$. Then $b = 8$ and $c = -4$.

Case 2: Let $a = -1$. Then $b = -8$ and $c = 4$.

Taking the standard solution ($a = 1, b = 8, c = -4$):

Values: $a = 1, b = 8, c = -4$.

Finding A^{-1}: For an orthogonal matrix, $A^{-1} = A^T$.

$$A^{-1} = \frac{1}{9}[8\ 1\ 4\ -4\ 4\ 7\ 1\ 8\ -4]$$

6. Find non-singular matrices P & Q such that PAQ is in normal form & hence find rank of the matrix A and obtain A^{-1} for $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$.

Sol:

Solution: We reduce A to Normal Form (I_3) using row operations (recorded in P) and column operations (recorded in Q). Start with: $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Step 1: $R_2 \to R_2 + R_1$

$$A \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix}, \quad P \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2: $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 + 2C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix}, \quad Q \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 3: Swap columns $C_2 \leftrightarrow C_3$ to get a simpler pivot (-2 is easier to work with than 5, or just to rearrange).

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 5 \\ 0 & 1 & -2 \end{bmatrix}, \quad Q \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Step 4: Swap rows $R_2 \leftrightarrow R_3$ to bring 1 to the pivot position.

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 5 \end{bmatrix}, \quad P \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

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Step 5: $R_3 \to R_3 + 2R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \quad P \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Step 6: $C_3 \to C_3 + 2C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Results:

1. **Normal Form:** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$
2. **Rank:** 3 (Number of non-zero rows in normal form).
3. **Matrix P:** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$
4. **Matrix Q:** $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$
5. **Inverse (A^{-1}):** Since $PAQ = I$, we have $A^{-1} = QP$.

$$A^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

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MODULE 6

1. Apply the Gauss Seidel method, to solve the following system of linear equations up to the two iterations. $x + 10y + z = 51, 15x + 3y - 2z = 85, x - 2y + 8z = 5$**.**

Sol:

Step 1: Rearrange for Diagonal DominanceTo ensure convergence, rearrange the equations so the largest coefficients are on the diagonal:

$15x + 3y - 2z = 85$ (Coefficient of x is 15)

$x + 10y + z = 51$ (Coefficient of y is 10)

$x - 2y + 8z = 5$ (Coefficient of z is 8)

Step 2: Iterative Formulas

$$x_{k+1} = \frac{1}{15}\left(85 - 3y_k + 2z_k\right)$$

$$y_{k+1} = \frac{1}{10}\left(51 - x_{k+1} - z_k\right)$$

$$z_{k+1} = \frac{1}{8}\left(5 - x_{k+1} + 2y_{k+1}\right)$$

Step 3: Iterations (Initial Guess: $x = 0, y = 0, z = 0$)

Iteration 1:

$x_1 = \frac{1}{15}(85 - 0 + 0) = 5.6667$

$y_1 = \frac{1}{10}(51 - 5.6667 - 0) = 4.5333$

$z_1 = \frac{1}{8}(5 - 5.6667 + 2(4.5333)) = 1.0500$

Iteration 2:

$x_2 = \frac{1}{15}(85 - 3(4.5333) + 2(1.0500)) = 4.9000$

$y_2 = \frac{1}{10}(51 - 4.9000 - 1.0500) = 4.5050$

$z_2 = \frac{1}{8}(5 - 4.9000 + 2(4.5050)) = 1.1388$

Answer :After 2 iterations, $x \approx 4.9000, y \approx 4.5050, z \approx 1.1388.$

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2. Apply Gauss-Seidel iteration method to solve the following linear equations up to the three iterations. $20x + y - 2z = 17, 3x + 20y - z = -18, 2x - 3y + 20z = 25$

Sol:

Sol. : We first write the equations as

$$x = \frac{1}{20}[17 - y + 2z] \quad \ldots\ldots\ldots\ldots (1)$$

$$y = \frac{1}{20}[-18 - 3x + z] \quad \ldots\ldots\ldots\ldots (2)$$

$$z = \frac{1}{20}[25 - 2x + 3y] \quad \ldots\ldots\ldots\ldots (3)$$

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(i) First Iteration : We start with the approximation $y = 0$, $z = 0$ and then get form (1).

$$x_1 = \frac{17}{20} = 0{\cdot}85$$

We use this approximation to find y *i.e.* we put $x = 0{\cdot}85$, $z = 0$ in (2)

$$\therefore \quad y_1 = \frac{1}{20}[-18 - 3(0{\cdot}85) - 0] = -1{\cdot}0275$$

We use these values of x_1 and y_1 to find z_1 *i.e.* we put $x = 0{\cdot}85$, $y = -1{\cdot}0275$ in (3).

$$\therefore \quad z_1 = \frac{1}{20}[25 - 2(0{\cdot}85) + 3(-1{\cdot}0275)] = 1{\cdot}0109$$

(ii) Second Iteration : We use latest values of y and z to find x *i.e.* we put $y = -1{\cdot}0275$, $z = 1{\cdot}0109$ in (1).

$$\therefore \quad x_2 = \frac{1}{20}[17 - (-1{\cdot}0275) + 2(1{\cdot}0109)] = 1{\cdot}0025$$

We put $x = 1{\cdot}0025$, $z = 1{\cdot}0109$ in (2).

$$\therefore \quad y_2 = \frac{1}{20}[-18 - 3(1{\cdot}0025) + 1{\cdot}0109] = -0{\cdot}9998$$

We put $x = 1{\cdot}0025$, $y = -0{\cdot}9998$ in (3).

$$\therefore \quad z_2 = \frac{1}{20}[25 - 2(1{\cdot}0025) + 3(-0{\cdot}9998)] = 0{\cdot}9998$$

(iii) Third Iteration : We use latest values of y and z to find x *i.e.* we put $y = -0{\cdot}9998$, $z = 0{\cdot}9998$ in (1).

$$\therefore \quad x_3 = \frac{1}{20}[17 - (-0{\cdot}9998) + 2(0{\cdot}9998)] = 1{\cdot}000$$

We put $x = 1{\cdot}000$, $z = 0{\cdot}9998$ in (2).

$$\therefore \quad y_3 = \frac{1}{20}[-18 - 3(1{\cdot}000) + 0{\cdot}9998] = -1{\cdot}000$$

We put $x = 1{\cdot}000$, $y = -1{\cdot}000$ in (3).

$$\therefore \quad z_3 = \frac{1}{20}[25 - 2(1{\cdot}000) + 3(-1{\cdot}000)] = 1$$

Hence, we get $x = 1$, $y = -1$, $z = 1$.

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3. Using Newton-Raphson Method, find the real root of $x^3 - 2x - 5 = 0$ correct to three decimal places.

Sol:

Sol. : Since $f(x) = x^3 - 2x - 5$; $f(0) = -5 < 0$, $f(1) = -6 < 0$, $f(2) = -1 < 0$, $f(3) = 16 > 0$.

Since $f(x)$ changes its sign from negative to positive as x goes from 2 to 3, there is a root between 2 and 3.

By Newton-Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2} = \frac{2x_n^3 + 5}{3x_n^2 - 2}$$

Since $f(2)$ is nearer to zero than $f(3)$, we start with $x_0 = 2$.

For $x_0 = 2$, $\quad x_1 = \dfrac{2(2)^3 + 5}{3(2)^2 - 2} = 2\cdot1$

For $x_1 = 2\cdot1$, $\quad x_2 = \dfrac{2(2\cdot1)^3 + 5}{3(2\cdot1)^2 - 2} = \dfrac{23\cdot522}{11\cdot23} = 2\cdot0946$

For $x_2 = 2\cdot0946$, $\quad x_3 = \dfrac{2(2\cdot0946)^3 + 5}{3(2\cdot0946)^2 - 2} = \dfrac{23\cdot3795}{11\cdot1620} = 2\cdot0946$

Hence, $x = 2\cdot0946$.

4. Using Newton Raphson's Method... find an iterative formula for $\sqrt[3]{150}$.

Sol:

Sol. : Let $x = \sqrt[3]{150}$ $\quad \therefore\ x^3 = 150$ $\quad \therefore\ x^3 - 150 = 0$

By Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad \text{Let } f(x) = x^3 - 150.$$

Iterative Formula : $\quad x_{n+1} = x_n - \dfrac{x_n^3 - 150}{3x_n^2} = \dfrac{2x_n^3 + 150}{3x_n^2}$

We start with $x_0 = 5$.

For $x_0 = 5$, $\quad x_1 = \dfrac{2(5)^3 + 150}{3(5)^2} = \dfrac{400}{75} = 5\cdot3333$

For $x_1 = 5\cdot3333$, $\quad x_2 = \dfrac{2(5\cdot3333)^3 + 150}{3(5\cdot3333)^2} = \dfrac{453\cdot4017}{85\cdot3323} = 5\cdot3134$

For $x_2 = 5\cdot3134$, $\quad x_3 = \dfrac{2(5\cdot3134)^3 + 150}{3(5\cdot3134)^2} = \dfrac{450\cdot0181}{84\cdot6966} = 5\cdot3133$

$\therefore\ \sqrt[3]{150} = 5\cdot3133$

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